42

$$y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4} & x \le 1 \\ x^3 - 6x^2 + 8x & x > 1 \end{cases}$$

$y' = -\frac{1}{2}x - \frac{1}{2}$

$y' = 3x^2 - 12x + 8$

$\frac{dy}{dx}\Big|_{x=1} \; -\frac{1}{2}(1) - \frac{1}{2} = -1$

$\frac{dy}{dx}\Big|_{x=1} \; 3(1)^2 - 12x + 8 = -1$

derivative defined everywhere

$-\frac{1}{2}x - \frac{1}{2} = 0$

$+\frac{1}{2} \quad +\frac{1}{2}$

$-\frac{1}{2}x = \frac{1}{2}$

$\boxed{x = -1}$

$3x^2 - 12x + 8 = 0$

$$\frac{12 \pm \sqrt{(-12)^2 - 4(3)(8)}}{2(3)}$$

$$\frac{12 \pm \sqrt{144 - 96}}{6} = \frac{12 \pm \sqrt{48}}{6}$$

$\sqrt{3} \cdot \sqrt{16}$

$$\frac{12 \pm 4\sqrt{3}}{6} = \frac{6 \pm 2\sqrt{3}}{3}$$

$$\boxed{2 \pm \frac{2\sqrt{3}}{3}}$$

$f(-1) = 4$ (max)

$f\left(2 + \frac{2\sqrt{3}}{3}\right) \approx -3.079$ (min)

29

$$y = \frac{x}{x^2+1} \qquad y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$y' = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

extrema can occur where y' is undefined, but

y' defined everywhere

b/c $x^2 \neq -1$ so bottom never 0

set $y'=0$ to find extrema

$$\cancel{(x^2+1)^2}\,\frac{-x^2+1}{\cancel{(x^2+1)^2}} = 0 \cdot (x^2+1)^2$$

$$-x^2+1=0$$

$$x^2=1$$

$$\underline{\underline{x=\pm 1}}$$

$f(1) = \frac{1}{2}$ (max)

$f(-1) = -\frac{1}{2}$ (min)

$(1, \frac{1}{2})$ max

$(1, -\frac{1}{2})$ min

16

$y = \sec x \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$

$y' = \sec x \tan x$

sec x undefined when $\cos x = 0$, $\therefore$ $\underline{\underline{x = \frac{\pi}{2}}}$ → Not a critical b/c the original function not defined at $x = \frac{\pi}{2}$

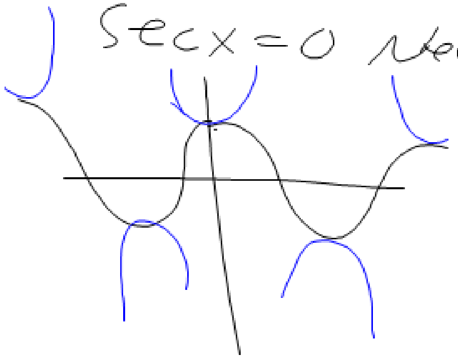
tan x undefined at $\frac{\pi}{2}$, or when $\cos x = 0$

$(\sec x)(\tan x) = 0$

$\tan x = 0$ when $\sin x = 0$

$\underline{\underline{x = 0, \pi}}$

sec x = 0 Never



22

$y = x^3 - 3x^2 + 3x - 2$

$y' = 3x^2 - 6x + 3$

$y' = 3(x^2 - 2x + 1)$

$y' = 3(x-1)^2$

critical point

$f'(x) = 0$

$f'(x) = $ undef.

$0 = 3(x-1)^2$

$x = 1$ makes $f'(x) = 0$

$\downarrow$

Nothing

$f(1) = 1 - 3 + 3 - 2$

$f(1) = {}^-1$

$f(2) = 8 - 12 + 6 - 2 = 0$

$f(0) = 0 - 0 + 0 - 2 = {}^-2$

37

$y = x\sqrt{4-x^2}$ $\qquad y' = x \cdot \frac{1}{\cancel{2}}(4-x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x + \sqrt{4-x^2} \cdot 1$

$y = x(4-x^2)^{\frac{1}{2}}$ $\qquad y = \dfrac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} \cdot \dfrac{\sqrt{4-x^2}}{\sqrt{4-x^2}}$

$$y = \frac{-x^2+4-x^2}{\sqrt{4-x^2}} \Rightarrow y = \frac{-2x^2+4}{\sqrt{4-x^2}}$$

$\sqrt{4-x^2}$

$$0 = \frac{-2x^2+4}{\cancel{\sqrt{4-x^2}}} \cdot \cancel{\sqrt{4-x^2}}$$

$$0 = -2x^2+4$$

$-4 \qquad\qquad -4$

$-4 = -2x^2$

$\boxed{x = \pm\sqrt{2} \quad , 2 + -2}$

$f(x) = x\sqrt{4-x^2}$ $\qquad \dfrac{\text{Domain}}{[-2, 2]}$

$f(\sqrt{2}) = \sqrt{2} \cdot \sqrt{4-\sqrt{2}^2}$

$= \sqrt{2} \cdot \sqrt{2}$

$f(\sqrt{2}) = 2$ (max)

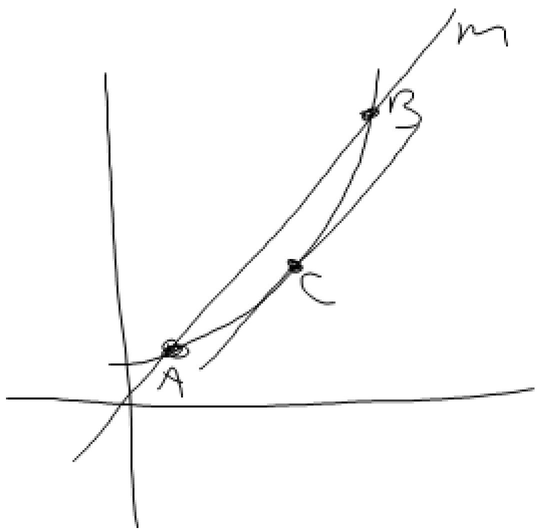
$f(-\sqrt{2}) = -2$ (min)

$f(2) = 0$ (rel. min)

$f(-2) = 0$ (rel. max)

Mean Value Theorem

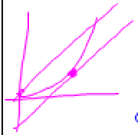
"If $y=f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b), then there is at least one point c in (a,b) at which $f'(c) = \frac{f(b)-f(a)}{b-a}$."



tangent

secant line

① Show that $f(x)=x^2$ satisfies the mean value thrm. on the interval $[0,2]$ and find pt. c.



continuous on $[0,2]$ yes

differentiable $(0,2)$ yes

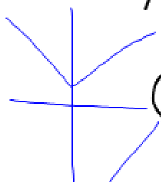
slope of secant line $\frac{f(2)-f(0)}{2-0}=\frac{4}{2}=2$

slope of tangent $y=2x$

$2x=2$

$\boxed{x=1}$

② Explain why each of the following fails the mean value thrm. on the interval $[-1,1]$



ⓐ $f(x)=\sqrt{x^2}+1$ Not differentiable at $x=0$ corner

ⓑ $f(x)=\begin{cases} x^3+3, & x<1 \\ x^2+1, & x\geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x)=(1)^3+3=4$

$\lim_{x\to 1^+} f(x)=(1)^2+1=2$

don't match

jump discontinuity

$y'=\frac{2x}{2\sqrt{x^2}}$

~~lft hand derivative as $x\to 0$ will be neg. b/c top will be neg and bottom pos.~~

~~rt hand deriv. as $x\to 0$ will pos. b/c both top & bottom will be pos.~~

derivative at 0 undefined

Corollary 1:

"If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b)

1. If $f'(x) > 0$ at every pt (a, b) then $f(x)$ increases over $[a, b]$
2. If $f'(x) < 0$, decreases

where does $f(x) = x^3 - 4x$ increase, decrease?

$$f'(x) = 3x^2 - 4$$

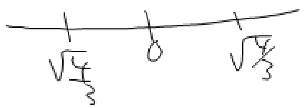
Increasing

$$3x^2 - 4 > 0$$
$$+4 \quad +4$$

$$\frac{3x^2}{3} > \frac{4}{3}$$

$$x^2 > 4/3$$

$$x \quad \pm\sqrt{\frac{4}{3}} \longrightarrow \quad x < -\sqrt{\frac{4}{3}} \quad x > \sqrt{\frac{4}{3}}$$



Decreasing

$$x^2 < \frac{4}{3}$$

$$x \quad \pm\sqrt{4/3} \quad -\sqrt{4/3} < x < \sqrt{4/3}$$

4.2 #2, 4, 7, 10, 11, 14, 15, 18, 22, 25, 28